Sydney Grammar School

NAME

MATHS MASTER

CANDIDATE NUMBER

2024 **Trial Examination**

Form VI Mathematics Extension 1

Friday 16th August, 2024
12.50 pm

General Instructions

- Reading time — 10 minutes
- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Fourteen Questions — 70 Marks

Section I (10 marks) Questions 1 – 10

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (60 marks) Questions 11 – 14

- Each question is worth 15 marks.
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Your name and master should only be written on this page.
- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.

Checklist

- Reference sheet
- Multiple-choice answer sheet
- 4 booklets per boy
- Candidature: 126 pupils

Writer: NJL

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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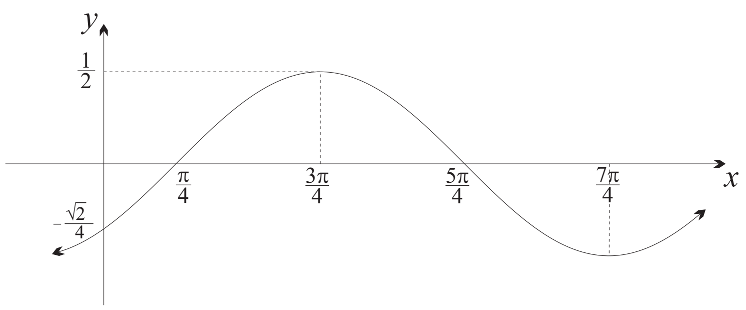
1. Given that there are 2000 employees in a company, which of the following represents the least number of employees that must share the same birthday?
   (A) 3
   (B) 4
   (C) 5
   (D) 6

2. What is the amplitude of $y = \sin x - \cos x$?
   (A) $-\sqrt{2}$
   (B) 1
   (C) $\sqrt{2}$
   (D) 2

3. Which of the following equations corresponds to the graph shown below?



   (A) $y = \frac{1}{2}\sin(x + \frac{\pi}{4})$
   (B) $y = \frac{1}{2}\sin(x - \frac{\pi}{4})$
   (C) $y = \frac{1}{2}\cos(x - \frac{\pi}{4})$
   (D) $y = \frac{1}{2}\cos(x + \frac{\pi}{4})$

4. Given that $\sin\theta = \dfrac{9}{41}$, and θ is obtuse, which of the fractions shown below is equivalent to $\sin 2\theta$?
   (A) $-\dfrac{720}{1681}$
   (B) $-\dfrac{81}{1681}$
   (C) $\dfrac{81}{1681}$
   (D) $\dfrac{720}{1681}$

5. Which of the following expressions is equal to $\dfrac{d}{dx}\tan^{-1}\sqrt{1-x}$?

   (A) $\dfrac{1}{2\sqrt{1-2x}}$

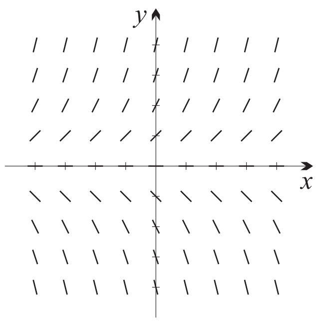
   (B) $-\dfrac{1}{2\sqrt{1-2x}}$

   (C) $\dfrac{1}{2\sqrt{1-x}}\sec^2\sqrt{1-x}$

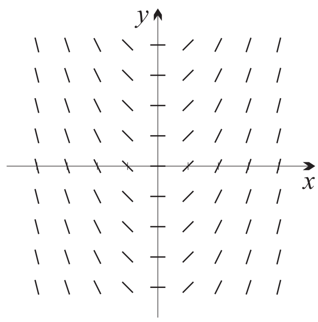
   (D) $\dfrac{1}{2(x-2)\sqrt{1-x}}$

6. Which of the following slope fields corresponds to the differential equation $y' = y - x$?

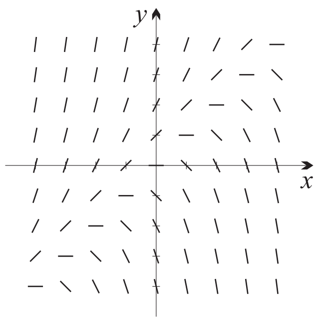
(A)



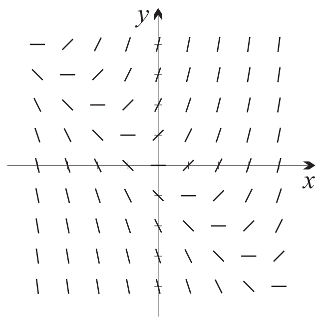
(B)



(C)



(D)



7. Given that $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -4\underset{\sim}{i} + \underset{\sim}{j}$, which of the following expressions represents the unit vector in the direction of $\underset{\sim}{a} + \underset{\sim}{b}$?

   (A) $-\frac{1}{\sqrt{2}}(\underset{\sim}{i} + \underset{\sim}{j})$

   (B) $\frac{1}{\sqrt{5}}(\underset{\sim}{i} + 2\underset{\sim}{j})$

   (C) $-\frac{1}{\sqrt{5}}(\underset{\sim}{i} + 2\underset{\sim}{j})$

   (D) $\frac{1}{\sqrt{2}}(\underset{\sim}{i} + \underset{\sim}{j})$

8. Which equation below is the general solution to $x + yy' = 0$, where C is an arbitrary constant and $y \neq 0$?

   (A) $y = -\frac{x^2}{2y} + C$

   (B) $y = -\frac{x^2}{y^2} + C$

   (C) $y = \pm\sqrt{C + x^2}$

   (D) $y = \pm\sqrt{C - x^2}$

9. Which of the following expressions is $\displaystyle\int \frac{4}{x}\big(\ln x\big)^3\,dx$?

   (A) $\big(\ln x\big)^4 + C$

   (B) $4x^4 \ln x + C$

   (C) $\dfrac{4x^4}{\ln x} + C$

   (D) $\dfrac{4}{x^4} + C$

10. Which differential equation below does not represent a form of the logistic equation?

   (A) $\frac{dN}{dt} = \frac{t}{50}\left(1 - \frac{t}{100}\right)$

   (B) $\frac{1}{N} \cdot \frac{dN}{dt} = 12 - \frac{N}{6}$

   (C) $\frac{dN}{dt} = \sqrt{3}\left(N - \left(\frac{N}{100}\right)^2\right)$

   (D) $\frac{dN}{dt} = \frac{N}{18}\left(2 - \frac{N}{10}\right)$

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet.

(a) (i) Find the values of a and b if $x^2 + 4x - 21 = (x+a)^2 - b$. **1**

(ii) Hence find $\displaystyle\int \frac{1}{\sqrt{21 - 4x - x^2}}\, dx$. **2**

(b) Let $f(x) = (x^2 + k)(2x+3) + 3$, where k is a constant.

(i) Write down the remainder when $f(x)$ is divided by $(2x+3)$. **1**

(ii) Given that the remainder when $f(x)$ is divided by $(x-2)$ is 24, prove that $k = -1$. **2**

(iii) Hence factorise $f(x)$ completely. **1**

(c) A school play requires at least two boys and exactly twice as many girls as boys. If four boys and six girls audition, how many different casts can be formed? **2**

(d) Find the coefficient of x^4 in the expansion of $(1+3x)^4(1-x)^5$. **3**

(e) Use the substitution $u = x^2 - 3x + 1$ to evaluate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\, dx$. Give your answer as an exact value. **3**

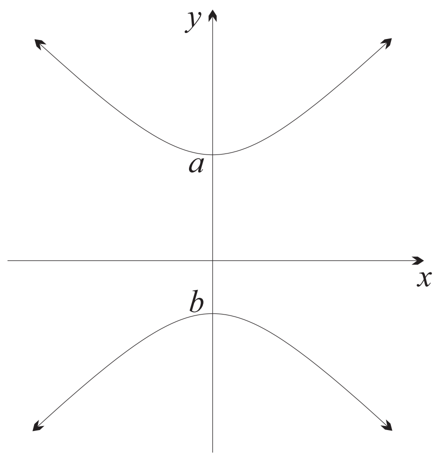
QUESTION TWELVE (15 marks) Start a new answer booklet.

(a) Let $f(x) = \cos^{-1} x$. Sketch the graphs of the following, clearly showing intercepts and endpoints. Your graphs should be on separate number planes and be about one-third of a page each.

(i) $y = f(x)$ **1**

(ii) $|y| = f(|x|)$ **1**

(b) The graph below has y-intercepts a and b and is defined by the parametric equations $x = 3\tan\theta$ and $y = 1 - 3\sec\theta$.



(i) Find the Cartesian equation of the graph. **2**

(ii) Hence, or otherwise, find the values of a and b. **2**

(c) At time $t = 0$, a football is kicked and leaves the ground with speed $20\,\text{m/s}$ at an angle of projection of $30°$ to the horizontal. Assume that upwards and to the right are positive and that the magnitude of acceleration due to gravity is $10\,\text{m/s}^2$.

(i) If the initial velocity is given by the vector $\underset{\sim}{V} = a\underset{\sim}{i} + b\underset{\sim}{j}$, find the values of a and b. **2**

(ii) How high is the ball above the ground when it has travelled $5\sqrt{3}$ metres horizontally? **2**

(iii) What is the exact speed of the ball, and the angle of the ball's velocity to the horizontal, when $t = 1.5\,\text{s}$? **2**

(d) Let $\underset{\sim}{a} = \lambda\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} - 2\underset{\sim}{j}$. The length of the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $3\sqrt{10}$. **3**

Find the possible values of λ.

QUESTION THIRTEEN (15 marks) Start a new answer booklet.

(a) Find the exact volume of revolution when the region bounded by the curve $y = \cos x$ and the x-axis, between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$, is rotated about the x-axis. **3**

(b) A spherical water purification tablet is added to a tank of water. After t hours, the tablet is a sphere with radius r mm, surface area $A\,\text{mm}^2$ and volume $V\,\text{mm}^3$. The tablet dissolves with a rate of change of volume directly proportional to its surface area, such that

$$\frac{dV}{dt} = -k_1 A,$$

where k_1 is a positive constant. Let V_0 be the volume at $t = 0$.

You may assume the formulae $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$.

(i) Show that $\dfrac{dV}{dt} = -k_2 V^{\frac{2}{3}}$, where $k_2 = k_1(36\pi)^{\frac{1}{3}}$. **2**

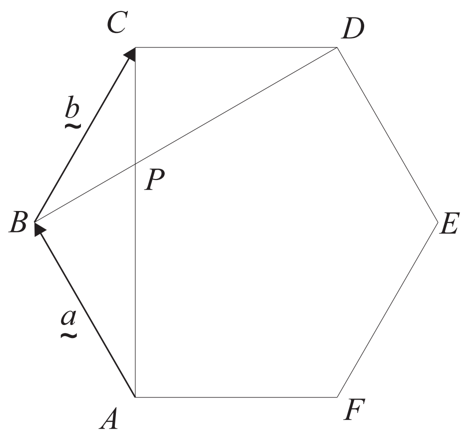
(ii) Solve $\dfrac{dV}{dt} = -k_2 V^{\frac{2}{3}}$ to obtain V in terms of t, k_2 and V_0. **2**

(iii) In six hours $\frac{7}{8}$ of the volume of the tablet has dissolved. Express k_2 in terms of V_0. **2**

(iv) How long, to the nearest minute, does it take for 99% of the volume of the tablet to dissolve? **1**

(c) The diagram below shows the regular hexagon $ABCDEF$. Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$.

Interval BD intersects AC at P where $\overrightarrow{AP} = \lambda\overrightarrow{AC}$.



(i) Express the ratio $|\overrightarrow{AP}| : |\overrightarrow{PC}|$ in terms of λ. **1**

(ii) Find an expression for $\overrightarrow{AP}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and λ. **1**

(iii) Use vector methods and the fact that $ABCDEF$ is a regular hexagon, to express $\overrightarrow{BD}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(iv) Hence use vector methods to find the exact value of λ. **2**

QUESTION FOURTEEN (15 marks) Start a new answer booklet.

(a) A drone flies over a city at a constant altitude of 800 m and velocity of 108 km/h. A soldier on the ground waits until the drone is directly above her before firing a bullet that leaves the gun at a speed of 300 m/s. You may assume that the magnitude of the acceleration due to gravity is $10\,\text{m/s}^2$.

(i) At what angle to the horizontal must she fire the bullet in order to hit the drone? **2**

(ii) How high above the ground must the drone be so that it is too high to shoot down? **3**

(b) The rate of change of a tadpole population is given by the differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$$

where P is the number of tadpoles after t days, and r and k are positive constants. Initially, the pond is home to P_0 tadpoles.

(i) Show that the tadpole population is increasing when $0 < P < k$. **1**

(ii) Solve the differential equation to find P as a function of t. **3**

You may assume $\dfrac{k}{P(k-P)} = \dfrac{1}{P} + \dfrac{1}{k-P}$.

(iii) Find $\lim\limits_{t\to\infty} P$, which is the maximum number of tadpoles that the pond can support. **1**

(iv) Find an expression for the time when the rate of change of population is greatest. **2**

(c) Consider the sequence $T_n = \sin\bigl(x + (n-1)y\bigr)$ for integers $n \geq 1$. **3**

Use mathematical induction to show that, for $n \geq 1$,

$$T_1 + T_2 + T_3 + \dots + T_n = \frac{\sin\bigl(x + \frac{1}{2}(n-1)y\bigr)\sin(\frac{1}{2}ny)}{\sin\frac{1}{2}y}\,.$$

END OF PAPER

Sydney Grammar School
Extension I Trial Examination 2024

1.) 2000 employees, 365 days.

$5 \times 365 = 1825$

$6 \times 365 = 2190$

$\therefore$ least number of employees that share the same birthday is 6.

D

2.) $y = \sin x - \cos x = R\cos(x - \alpha)$

$R = \sqrt{(1)^2 + (-1)^2}$

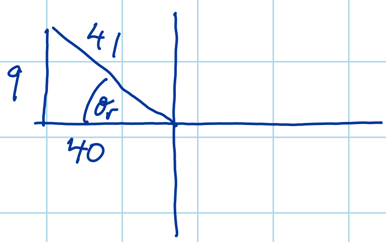
$= \sqrt{2}$

C

3.) $y = \frac{1}{2}\sin(x - \pi/4)$

B

4.) $\sin\theta = \frac{9}{41}$ $90° < \theta < 180°$ $\therefore$ 2nd quadrant.



$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = \frac{9}{41}$ $\cos\theta = -\frac{40}{41}$

$\sin 2\theta = 2 \times \frac{9}{41} \times -\frac{40}{41}$

$= -\frac{720}{1681}$

A

5.) $\frac{d}{dx} \tan^{-1}\sqrt{1-x}$

$u = (1-x)^{1/2}$

$u' = \frac{1}{2}(1-x)^{-1/2}$

$= \frac{1}{2\sqrt{1-x}}$

$\frac{d}{dx} \tan^{-1} u = \frac{u'}{1+u^2}$

$= \frac{\frac{-1}{2\sqrt{1-x}}}{1+[\sqrt{1-x}]^2}$

$= \frac{-1}{2\sqrt{1-x}(1+1-x)}$

$= \frac{-1}{2(2-x)\sqrt{1-x}}$

$= \frac{1}{2(x-2)\sqrt{1-x}}$

D

6.) $y' = y - x$

C

7.) $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ $\underset{\sim}{a} + \underset{\sim}{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\therefore \frac{1}{2\sqrt{2}}(-2\underset{\sim}{i} - 2\underset{\sim}{j})$

$= -\frac{2}{2\sqrt{2}}(\underset{\sim}{i} + \underset{\sim}{j})$

$= -\frac{1}{\sqrt{2}}(\underset{\sim}{i} + \underset{\sim}{j})$

A

8.) $x + yy' = 0 \qquad y \neq 0$

$yy' = -x$

$y\frac{dy}{dx} = -x$

$\int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + C$

$y^2 = -x^2 + C$

$y = \pm\sqrt{C - x^2}$

D

9.) $\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + C$

$4\int \frac{1}{x}\left[\ln(x)\right]^3 dx = 4 \times \frac{1}{4}\left[\ln(x)\right]^4 + C$

$= (\ln(x))^4 + C$

A

10.) $\frac{dN}{dt} = \frac{t}{50}\left(1 - \frac{t}{100}\right)$ is not of the form

$\frac{dN}{dt} = KN(P - N)$

A

11) a) i) $x^2+4x-21$

$= (x+2)^2-4-21$

$= (x+2)^2-25$ ✓

$a=2 \qquad b=25$

ii) $\displaystyle\int \frac{1}{\sqrt{-(x^2+4x-21)}}\,dx$

$\displaystyle= \int \frac{1}{\sqrt{-[(x+2)^2-25]}}\,dx$

$\displaystyle= \int \frac{1}{\sqrt{25-(x+2)^2}}\,dx$ ✓

$\displaystyle= \sin^{-1}\frac{x+2}{5} + C$ ✓

Working mark for recognising $\sin^{-1}$ of some function.

b) i) $\dfrac{(x^2+k)(2x+3)+3}{2x+3} = x^2+k$ remainder 3. ✓

ii) $f(2) = 24$ ✓

$24 = (4+k)(4+3)+3$

$24 = (4+k)\times 7 + 3$

$21 = 7(4+k)$

$3 = 4+k$

$\underline{k=-1}$ ✓

iii) $f(x) = (x^2-1)(2x+3)+3$

$= 2x^3+3x^2-2x-3+3$

$= 2x^3+3x^2-2x$

$= x(2x^2+3x-2)$

$= x(2x-1)(x+2)$ ✓

c) At least 2 boys, exactly twice as many girls.

boys	girls
2	4
3	6

for either case identified.

$\underbrace{{}^4C_2 \times {}^6C_4}_{\text{either}} + \underbrace{{}^4C_3 \times {}^6C_6}_{\text{either}} = 94$ ✓

either ✓

d) $(1+3x)^4 = {}^4C_0 + {}^4C_1(3x) + {}^4C_2(3x)^2 + {}^4C_3(3x)^3 + {}^4C_4(3x)^4$

$(1-x)^5 = {}^5C_0 + {}^5C_1(-x) + {}^5C_2(-x)^2 + {}^5C_3(-x)^3 + {}^5C_4(-x)^4 + {}^5C_5(-x)^5$

terms in x^4:

$${}^4C_0 \times {}^5C_4(-x)^4 + {}^4C_1(3x) \times {}^5C_3(-x)^3 + {}^4C_2(3x)^2 \times {}^5C_2(-x)^2 + {}^4C_3(3x)^3 \times {}^5C_1(-x) + {}^4C_4(3x)^4 \times {}^5C_0$$

✓ for any one of correct 5 cases seen, above.

$$= 1 \times 5x^4 + 12x(-10x^3) + 54x^2(10x^2) + 108x^3(-5x) + 81x^4 \quad ✓$$

$$= 5x^4 - 120x^4 + 540x^4 - 540x^4 + 81x^4$$

coefficient $= \underline{\underline{-34}}$ ✓

e) $$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$$

$u = x^2-3x+1$

$\frac{du}{dx} = 2x-3$

$\frac{du}{2x-3} = dx$

$5 \rightarrow 11$

$3 \rightarrow 1$

$$= \int_1^{11} \frac{2x-3}{\sqrt{x^2-3x+1}} \frac{du}{(2x-3)}$$

$$= \int_1^{11} u^{-1/2}\,du \quad ✓ \text{ or equival. limits req.}$$

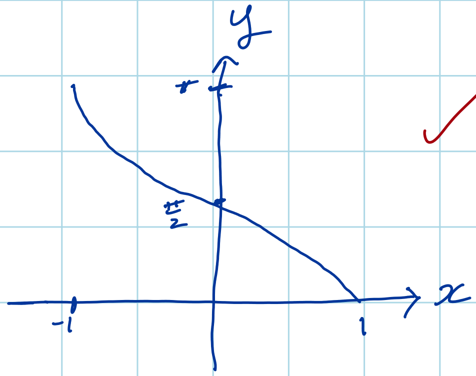
$$= 2\left[u^{1/2}\right]_1^{11} \quad ✓$$

$$= 2(\sqrt{11}-1) \quad ✓$$

(factorisation not req.)

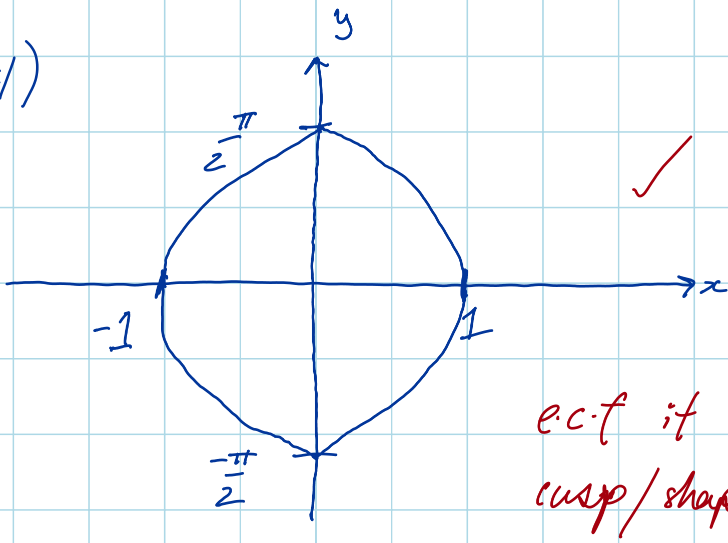
12) a) i)

$y = \cos^{-1}(x)$



✓

ii) $|y| = \cos^{-1}(|x|)$



✓

e.c.f if i) incorrect.
cusp/shape

Don't deduct missing x and/or y on axes.
Do deduct for any missing values as per above.

b) i) $x = 3\tan\theta$

$x^2 = 9\tan^2\theta$ ✓

$x^2 = 9(\sec^2\theta - 1)$

$y = 1 - 3\sec\theta$

$3\sec\theta = 1 - y$

$9\sec^2\theta = (1-y)^2$

$9\sec^2\theta - 9 = (1-y)^2 - 9$

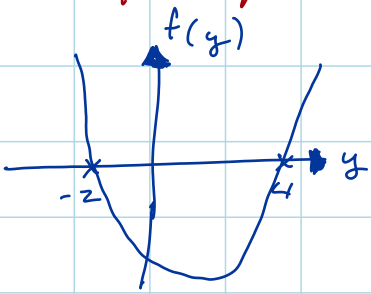
$9(\sec^2\theta - 1) = (1-y-3)(1-y+3)$

$x^2 = (-2-y)(4-y)$

$x^2 = (y+2)(y-4)$ ✓ (or equiv)

ii) $x = \pm\sqrt{(y+2)(y-4)}$

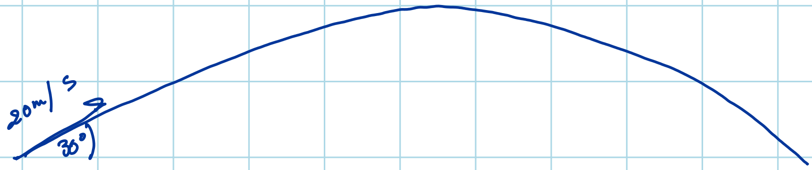
$(y+2)(y-4) \geq 0$



$y \geq 4$ or $y \leq -2$

$a = 4$, $b = -2$ ✓ (Both)

✓ Working mark if substitute $x=0$ into cartesian or parametric equations.

c) 

i) 

initially; $\underset{\sim}{V} = 20\cos 30\,\underset{\sim}{i} + 20\sin 30\,\underset{\sim}{j}$ ✓

$\underset{\sim}{V} = 10\sqrt{3}\,\underset{\sim}{i} + 10\underset{\sim}{j}$ ✓ (Both required for 2nd mark).

$a = 10\sqrt{3}$. $b = 10$

ii) $\dot{x} = 10\sqrt{3}$

$x = 10\sqrt{3}t$

Let $x = 5\sqrt{3}$

$5\sqrt{3} = 10\sqrt{3}t$

$t = \frac{1}{2}$ s. ✓

$\ddot{y} = -10$

$\dot{y} = -10t + c$

$t = 0,\ \dot{y} = 10,\ \therefore c = 10$

$\dot{y} = -10t + 10$

$y = -5t^2 + 10t + D$

$t = 0,\ y = 0,\ \therefore D = 0$

$y = -5t^2 + 10t$

at $t = \frac{1}{2}$

$y = -5\left(\frac{1}{2}\right)^2 + 10\left(\frac{1}{2}\right)$

$y = 3\frac{3}{4}$ m ✓

(units not required)

iii) $t = 1.5$ $\quad \dot{x} = 10\sqrt{3}$

$$\dot{y} = -10 \times 1.5 + 10$$

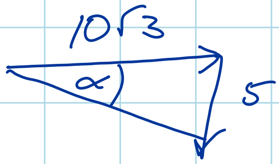
$$\dot{y} = -5$$

$$\text{speed} = \sqrt{(10\sqrt{3})^2 + (-5)^2}$$

$$= \sqrt{325} \text{ m/s}$$

$$= 5\sqrt{13} \text{ m/s}.$$

Let angle to horizontal be α.



$$|\tan \alpha| = \frac{5}{10\sqrt{3}}$$

$$\alpha = 16.102$$

$\therefore$ Angle to horizontal $= -16°$

(To nearest degree).

(must be negative, any reasonable rounding.)

d) $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}^2}\right)\underset{\sim}{b}$

$= \left(\frac{6\lambda - 6}{40}\right)\begin{bmatrix}6\\-2\end{bmatrix}$ ✓

$\text{length} = \left|\frac{6\lambda-6}{40}\right|\sqrt{40}$ ✓

$\text{length} = \frac{|6\lambda - 6|}{\sqrt{40}} = 3\sqrt{10}$

$|6\lambda - 6| = 60$

$6\lambda - 6 = 60$ or $6\lambda - 6 = -60$

$\lambda - 1 = 10$ $\qquad \lambda - 1 = -10$

$\lambda = 11$ $\qquad \lambda = -9$ ✓ (Both req. for last mark).

Alternatively:

$\underset{\sim}{a} = \lambda\underset{\sim}{i} + 3\underset{\sim}{j} \qquad \underset{\sim}{b} = 6\underset{\sim}{i} - 2\underset{\sim}{j}$

$|\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}| = \left|\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b}\right|$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \begin{bmatrix}\lambda\\3\end{bmatrix}\cdot\begin{bmatrix}6\\-2\end{bmatrix} = 6\lambda - 6$

$\underset{\sim}{b}\cdot\underset{\sim}{b} = \begin{bmatrix}6\\-2\end{bmatrix}\cdot\begin{bmatrix}6\\-2\end{bmatrix} = 36 + 4 = 40$

$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{6\lambda-6}{40} \times \begin{bmatrix}6\\-2\end{bmatrix}$ ✓ $\frac{3\lambda-3}{20}(6\underset{\sim}{i} - 2\underset{\sim}{j})$

$= \frac{3\lambda-3}{10}(3\underset{\sim}{i} - \underset{\sim}{j})$

$|\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}| = \left|\frac{3\lambda-3}{10}(3\underset{\sim}{i} - \underset{\sim}{j})\right|$

$= \left|\frac{3\lambda-3}{20} \times \sqrt{40}\right|$

$= \left|\frac{(3\lambda-3)\times 2\sqrt{10}}{20}\right|$

$= \left|\frac{(3\lambda-3)\sqrt{10}}{10}\right|$

$\left|\frac{(3\lambda-3)\sqrt{10}}{10}\right| = 3\sqrt{10}$ ✓

$\frac{(3\lambda-3)\sqrt{10}}{10} = 3\sqrt{10}$

$\frac{3\lambda-3}{10} = 3$

$\frac{\lambda-1}{10} = 1$

$\lambda - 1 = 10$

$\lambda = 11$

$\frac{(3\lambda-3)\cancel{\sqrt{10}}}{10} = -3\cancel{\sqrt{10}}$

$\frac{\lambda-1}{10} = -1$

$\lambda - 1 = -10$

$\lambda = -9$ (✓)

(Both req.)

13). a) $\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$ $\qquad \frac{\pi}{4} < \pi/3$

$$V = \int_{\pi/4}^{\pi/3} \pi y^2 \, dx = \pi \int_{\pi/4}^{\pi/3} \cos^2 x \, dx$$

$$= \pi \int_{\pi/4}^{\pi/3} \tfrac{1}{2}(1 + \cos 2x) \, dx \quad \checkmark$$

$$= \frac{\pi}{2} \int_{\pi/4}^{\pi/3} 1 + \cos 2x \, dx$$

$$= \frac{\pi}{2} \left[x + \frac{1}{2} \sin 2x \right]_{\pi/4}^{\pi/3} \quad \checkmark$$

$$= \frac{\pi}{2} \left[\left(\frac{\pi}{3} + \frac{1}{2} \sin(2\pi/3) \right) - \left(\pi/4 + \tfrac{1}{2} \sin(2\pi/4) \right) \right]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{3} + \frac{1}{2} \sin(\pi/3) - \pi/4 - \frac{1}{2} \sin(\pi/2) \right]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{12} + \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times 1 \right]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2} \right]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{12} + \frac{3\sqrt{3}}{12} - \frac{6}{12} \right]$$

$$= \frac{\pi}{24} \left(\pi + 3\sqrt{3} - 6 \right) \; u^3 \quad \checkmark$$

b) i) Show $\frac{dV}{dt} = -k_2 V^{2/3}$ where $k_2 = k_1 (36\pi)^{1/3}$

Given: $\frac{dV}{dt} = -k_1 A$

$A = 4\pi r^2$ $\quad V = \frac{4}{3}\pi r^3$

$$\frac{3V}{4\pi} = r^3$$

$$r = \left(\frac{3V}{4\pi}\right)^{1/3}$$

$$\therefore \frac{dV}{dt} = -k_1 \times 4\pi \left(\frac{3V}{4\pi}\right)^{2/3}$$

$$\frac{dV}{dt} = -k_1 \times \frac{4\pi}{(4\pi)^{2/3}} \times 3^{2/3} \times V^{2/3}$$

$$\frac{dV}{dt} = -k_1 \times (4\pi)^{1/3} \times 9^{1/3} \times V^{2/3}$$

$$\frac{dV}{dt} = -k_1 (36\pi)^{1/3} \times V^{2/3}$$

$$\therefore \frac{dV}{dt} = -k_2 V^{2/3} \quad \text{where } k_2 = k_1 (36\pi)^{1/3}$$

ii)

$$\frac{dV}{dt} = -k_2 V^{2/3}$$

$$\int V^{-2/3}\, dV = -\int k_2\, dt$$

$$3V^{1/3} = -k_2 t + c$$

at $t = 0$ $\quad V = V_0$

$$3V_0^{1/3} = c$$

$\therefore$

$$3V^{1/3} = -k_2 t + 3V_0^{1/3}$$

$$3V^{1/3} = 3V_0^{1/3} - k_2 t$$

$$V^{1/3} = V_0^{1/3} - \frac{1}{8} k_2 t$$

$$V = \left(V_0^{1/3} - \frac{1}{3} k_2 t\right)^3$$

iii) when $t = 6$ $\quad V = \frac{V_0}{8}$

$$\frac{V_0}{8} = \left(V_0^{1/3} - \frac{1}{3} \times k_2 \times 6\right)^3$$

$$\frac{V_0^{1/3}}{2} = V_0^{1/3} - 2k_2$$

$$2k_2 = \frac{V_0^{1/3}}{2}$$

$$k_2 = \frac{V_0^{1/3}}{4}$$

iv) 99% dissolved $\therefore$ $V = \frac{V_0}{100}$

$$V = \left(V_0^{1/3} - \frac{1}{3}k_2 t\right)^3 \qquad k_2 = \frac{V_0^{1/3}}{4}$$

$$\frac{V_0}{100} = \left(V_0^{1/3} - \frac{1}{3} \times \frac{V_0^{1/3}}{4} \times t\right)^3$$

$$\frac{V_0}{100} = \left(V_0^{1/3}\left(1 - \frac{t}{12}\right)\right)^3$$

$$\frac{V_0}{100} = V_0\left(1 - \frac{t}{12}\right)^3$$

$$\frac{1}{100} = \left(1 - \frac{t}{12}\right)^3$$

$$\frac{1}{\sqrt[3]{100}} = 1 - \frac{t}{12}$$

$$\frac{t}{12} = 1 - \frac{1}{\sqrt[3]{100}}$$

$$t = 12\left(1 - \frac{1}{\sqrt[3]{100}}\right)$$

$t \approx 9.414678\ldots$ hrs

$t = 9$ hours and 25 minutes. ✓

(Do not penalise rounding despite what the question says.)

c) i) $|\overrightarrow{AP}| : |\overrightarrow{PC}|$

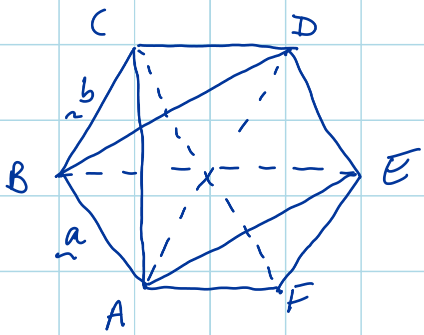
$\lambda : 1-\lambda$ ✓

ii) $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{AP} = \lambda \times \overrightarrow{AC}$

$= \lambda(\underset{\sim}{a} + \underset{\sim}{b})$ ✓

iii)



$\overrightarrow{BD} = \overrightarrow{AE}$

$= 2\underset{\sim}{b} - \underset{\sim}{a}$ ✓

iv) $\overrightarrow{BP} = (1-\lambda)\overrightarrow{BD}$

$= (1-\lambda)(2\underset{\sim}{b} - \underset{\sim}{a})$ ✓

$\therefore \overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$

$= \underset{\sim}{a} + (1-\lambda)(2\underset{\sim}{b} - \underset{\sim}{a})$

$= a + 2\underset{\sim}{b} - \underset{\sim}{a} - 2\lambda\underset{\sim}{b} + \lambda\underset{\sim}{a}$

$= \lambda\underset{\sim}{a} + (2-2\lambda)\underset{\sim}{b}$

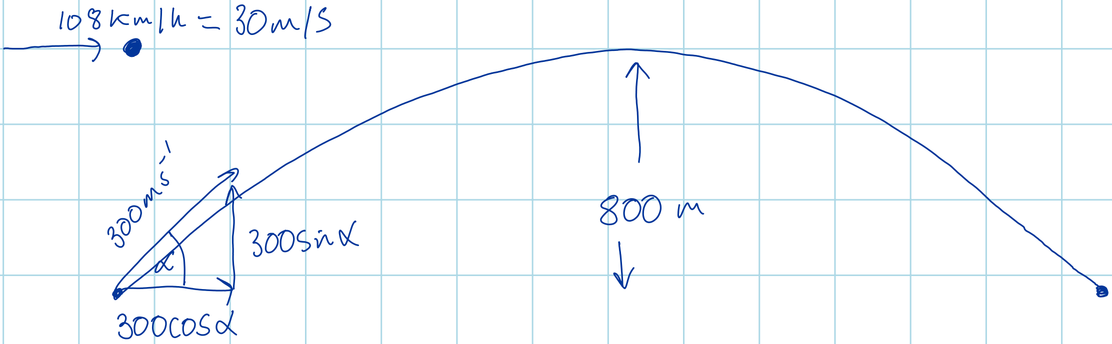
but from ii) $\overrightarrow{AP} = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$

Comparing coefficients of $\underset{\sim}{b}$: $\lambda = 2 - 2\lambda$

$3\lambda = 2$

$\lambda = 2/3$ ✓

14) a) i)



Let angle to horizontal be α.
To hit, horizontal displacement of both drone and bullet must be equal.

Drone: $\dot{x} = 30$
$x = 30t$

Bullet: $\ddot{x} = 0$
$\dot{x} = 300\cos\alpha$
$x = 300t\cos\alpha$

(both expressions for x.)

$\therefore$ $300t\cos\alpha = 30t$ to hit

$\cos\alpha = \frac{1}{10}$

$\alpha = \cos^{-1}\left(\frac{1}{10}\right)$

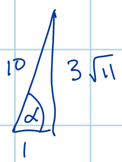
$\approx 84^\circ$ (to nearest degree)

ii) Bullet

Vertically: $\ddot{y} = -10$

$\dot{y} = -10t + 300\sin\alpha$

$y = -5t^2 + 300t\sin\alpha$



$\cos\alpha = \frac{1}{10}$

$\sin\alpha = \frac{3\sqrt{11}}{10}$

$y = -5t^2 + 300t\sin\alpha$

$y = -5t^2 + 300t\left(\frac{3\sqrt{11}}{10}\right)$

$y = -5t^2 + 90\sqrt{11}\,t$

Max ht when $\dot{y} = 0$ $\quad -10t + 300\sin\alpha = 0$

$t = 30\sin\alpha$

$= 30\,\frac{3\sqrt{11}}{10}$

$= 9\sqrt{11}$ secs

[or

Find when $y=0$, use symmetry to find when $y = y_{max}$

Let $y = 0$

$0 = -5t^2 + 90\sqrt{11}\, t$

$0 = t^2 - 18\sqrt{11}\, t$

$0 = t(t - 18\sqrt{11})$

$t = 0$ and $t = 18\sqrt{11}$ gives $y = 0$

y_{max} when $t = \frac{0 + 18\sqrt{11}}{2} = 9\sqrt{11}$ seconds]

$$y_{max} = -5\left(9\sqrt{11}\right)^2 + 90\sqrt{11}\left(9\sqrt{11}\right)$$

$$= -5 \times 81 \times 11 + 810 \times 11$$

$$= 4455 \text{ m}$$

$$= 4.455 \text{ km}$$

b) i)

Tadpoles: $0 < P < k$

$P < k$

$\frac{P}{k} < 1$

$0 < 1 - \frac{P}{k}$

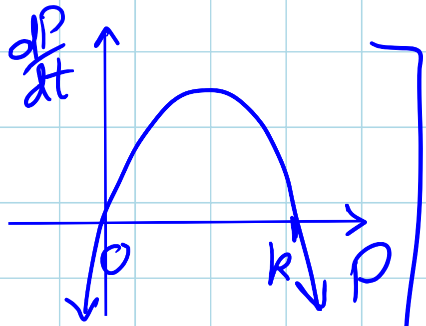
$1 - \frac{P}{k} > 0$

$\therefore \; rP\left(1 - \frac{P}{k}\right) > 0 \qquad r, P > 0$

$\therefore \; \frac{dP}{dt} > 0$

logical argument. ✓

OR $\left[\frac{dP}{dt} > 0 \text{ for } 0 < P < k \right.$



ii) $\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$

$k \cdot \frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)k$

$k \cdot \frac{dP}{dt} = rP(k - P)$

$\int \frac{k}{P(k-P)} \, dP = \int r \, dt$

Since $\frac{k}{P(k-P)} = \frac{1}{P} + \frac{1}{k-P}$

$\int \frac{1}{P} + \frac{1}{k-P} \, dP = rt + C_1$

$\ln|P| - \ln|k-P| = rt + C_1$ ✓

$\ln\left|\frac{P}{k-P}\right| = rt + C_1$

$-\ln\left|\frac{k-P}{P}\right| = rt + C_2$

$\ln\left|\frac{k}{P} - 1\right| = -rt + C_2$

$\frac{k}{P} - 1 = e^{-rt + C_2}$

$\frac{k}{P} = 1 + e^{-rt + C_2}$

Let $A = e^{C_2}$

$\frac{k}{P} = 1 + Ae^{-rt}$

$P = \frac{k}{1 + Ae^{-rt}}$ ✓

at $t = 0$, $P = P_0$

$P_0 = \frac{k}{1 + A}$

$1 + A = \frac{k}{P_0}$

$$A = \frac{K}{P_0} - 1$$

$$A = \frac{K}{P_0} - \frac{P_0}{P_0}$$

$$A = \frac{K - P_0}{P_0}$$

Requires clear, full, logical progression for 3 marks.

$$\therefore P = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right)e^{-rt}}$$

$$P = \frac{KP_0}{P_0 + (K-P_0)e^{-rt}} \quad \checkmark$$ as required.

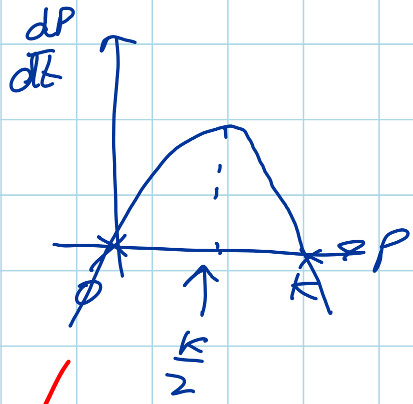
iii) $$\lim_{t\to\infty} P = \lim_{t\to\infty} \frac{KP_0}{P_0 + (K-P_0)e^{-rt}}$$

$$= \frac{KP_0}{P_0}$$

$$= K$$

✓ Needs justification

iv) $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$



$P = 0$ or $1 - \frac{P}{K} = 0$

$P = K$

maximum at $P = \frac{K}{2}$

$$P = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

when $P = \frac{K}{2}$

$$\frac{K}{2} = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

$$\frac{1}{2} = \frac{P_0}{P_0 + (K - P_0)e^{-rt}}$$

$$P_0 + (K - P_0)e^{-rt} = 2P_0$$

$$(K - P_0)e^{-rt} = P_0$$

$$e^{-rt} = \frac{P_0}{K - P_0}$$

$$-rt = \ln\left|\frac{P_0}{K - P_0}\right|$$

$$t = -\frac{1}{r}\ln\left|\frac{P_0}{K - P_0}\right|$$

$$t = \frac{1}{r}\ln\left|\frac{K - P_0}{P_0}\right|$$

c) Prove $\sin x + \sin(x+y) + \sin(x+2y) + \cdots + \sin(x+(n-1)y)$

$$= \frac{\sin\left(x + \frac{1}{2}(n-1)y\right)\sin\left(\frac{1}{2}ny\right)}{\sin\left(\frac{1}{2}y\right)}$$

for $n \geq 1$

A. Let $n = 1$

$LHS = \sin(x)$

$$RHS = \frac{\sin\left(x + \frac{1}{2} \times 0 \times y\right)\sin\left(\frac{1}{2}y\right)}{\sin\left(\frac{1}{2}y\right)}$$

$= \sin(x)$

$= LHS.$

The statement is true for $n = 1$.

B. Assume true for $n = k$. Assume

$$\sin x + \sin(x+y) + \sin(x+2y) + \cdots + \sin(x+(k-1)y)$$
$$= \frac{\sin\left(x + \frac{k-1}{2}y\right)\sin\left(\frac{1}{2}ky\right)}{\sin\left(\frac{1}{2}y\right)}$$

Now prove for $n = k+1$, prove

$$\sin(x) + \sin(x+y) + \sin(x+2y) + \cdots + \sin(x+(k-1)y) + \sin(x+ky)$$
$$= \frac{\sin\left(x + \frac{1}{2}ky\right)\sin\left(\frac{1}{2}(k+1)y\right)}{\sin\left(\frac{1}{2}y\right)}$$

Set up Correct

$LHS = \sin(x) + \sin(x+y) + \sin(x+2y) + \cdots + \sin(x+(k-1)y) + \sin(x+ky)$

$$= \frac{\sin\left(x + \frac{1}{2}(k-1)y\right)\sin\left(\frac{1}{2}ky\right)}{\sin\left(\frac{1}{2}y\right)} + \sin(x+ky)$$

* By the induction hypothesis.

$$= \frac{\sin\left(x+\frac{1}{2}(k-1)y\right)\sin\left(\frac{1}{2}ky\right)}{\sin\left(\frac{1}{2}y\right)} + \frac{\sin(x+ky)\sin\left(\frac{1}{2}y\right)}{\sin\left(\frac{1}{2}y\right)}$$

$$= \frac{\sin\left(x+\frac{1}{2}(k-1)y\right)\sin\left(\frac{1}{2}ky\right) + \sin(x+ky)\sin\left(\frac{1}{2}y\right)}{\sin\left(\frac{1}{2}y\right)}$$

$$\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$= \frac{\frac{1}{2}\left[\cos\left(x+\frac{k-1}{2}y-\frac{ky}{2}\right)-\cos\left(x+\frac{k-1}{2}y+\frac{ky}{2}\right)\right] + \frac{1}{2}\left[\cos\left(x+ky-\frac{y}{2}\right)-\cos\left(x+ky+\frac{y}{2}\right)\right]}{\sin\left(\frac{y}{2}\right)}$$

✓ Use of Product to Sums

$$= \frac{\frac{1}{2}\cos\left(x-\frac{y}{2}\right) - \frac{1}{2}\cos\left(x+y\left(\frac{2k-1}{2}\right)\right) + \frac{1}{2}\cos\left(x+y\left(\frac{2k-1}{2}\right)\right) - \frac{1}{2}\cos\left(x+y\left(\frac{2k+1}{2}\right)\right)}{\sin\left(\frac{y}{2}\right)}$$

$$= \frac{\frac{1}{2}\cos\left(x-\frac{y}{2}\right) - \frac{1}{2}\cos\left(x+\left(k+\frac{1}{2}\right)y\right)}{\sin \frac{y}{2}}$$

$$= \frac{\frac{1}{2}\left[\cos\left\{\left(x+\frac{ky}{2}\right)-\left(\frac{ky}{2}+\frac{y}{2}\right)\right\} - \cos\left\{\left(x+\frac{ky}{2}\right)+\left(\frac{ky}{2}+\frac{y}{2}\right)\right\}\right]}{\sin \frac{y}{2}}$$

✓ use of sum to product

$$= \frac{\sin\left(x+\frac{ky}{2}\right)\sin\left(\frac{ky}{2}+\frac{y}{2}\right)}{\sin \frac{y}{2}}$$

Using product to sums where $A = \left(x+\frac{ky}{2}\right)$
$B = \left(\frac{ky}{2}+\frac{y}{2}\right)$

$= RHS$

By Principle of Mathematical Induction result true for all positive integer n.

c. By the process of mathematical induction,
the statement is true for $n=1$,
$n=k$ and $n=k+1$, therefore
is true for all $n \geq 1$.